ALL SAINTS' COLLEGE

WA Exams Practice Paper B, 2016

Question/Answer Booklet

MATHEMATICS SPECIALIST UNITS 3 AND 4

Section One: Calculator-free

SOLUTIONS

Student Number: In figures | | | | | | | | |

In words _____

Your name _____

Time allowed for this section

Reading time before commencing work: five minutes
Working time for section: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
			Total	150	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

See next page

Section One: Calculator-free **35% (52 Marks)**

This section has **seven (7)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 **(6 marks)**

Let $g(z) = z^3 + (i-3)z^2 + (6-3i)z - 18$, $z \in \mathbb{C}$.

(a) Determine the remainder when $g(z)$ is divided by $2i$. (2 marks)

$$g(2i) = (2i)^3 + (i-3)(2i)^2 + (6-3i)(2i) - 18$$

$$= -8i - 4i + 12 + 12i + 6 - 18$$

$$= 0 \Rightarrow \text{remainder is 0.}$$

(b) Solve $g(z) = 0$. (4 marks)

$z - 2i$ is a factor

$$\frac{z^3 + (i-3)z^2 + (6-3i)z - 18}{z - 2i} = z^2 + (3i-3)z - 9i$$

$$z^2 + (3i-3)z - 9i = (z-3)(z+3i)$$

$$z = 3,\ 2i,\ -3i$$

Question 2 **(8 marks)**

(a) Determine $\int_0^2 \frac{x}{\sqrt{x^2+1}}\,dx$. (4 marks)

$$u = x^2 + 1 \Rightarrow du = 2x\,dx$$

$$x = 0,\ u = 1;\quad x = 2,\ u = 5$$

$$\int_0^2 \frac{1}{2}\frac{2x}{\sqrt{x^2+1}}\,dx = \int_1^5 \frac{1}{2\sqrt{u}}\,du$$

$$= \left[\sqrt{u}\right]_1^5$$

$$= \sqrt{5} - 1$$

(b) Determine $\int \left(4\cos^2\theta - 2\sin^2\theta - 1\right) d\theta$. (4 marks)

$$\int \left(4\cos^2\theta - 2\sin^2\theta - 1\right) d\theta = \int 3 - 6\sin^2\theta\ d\theta$$

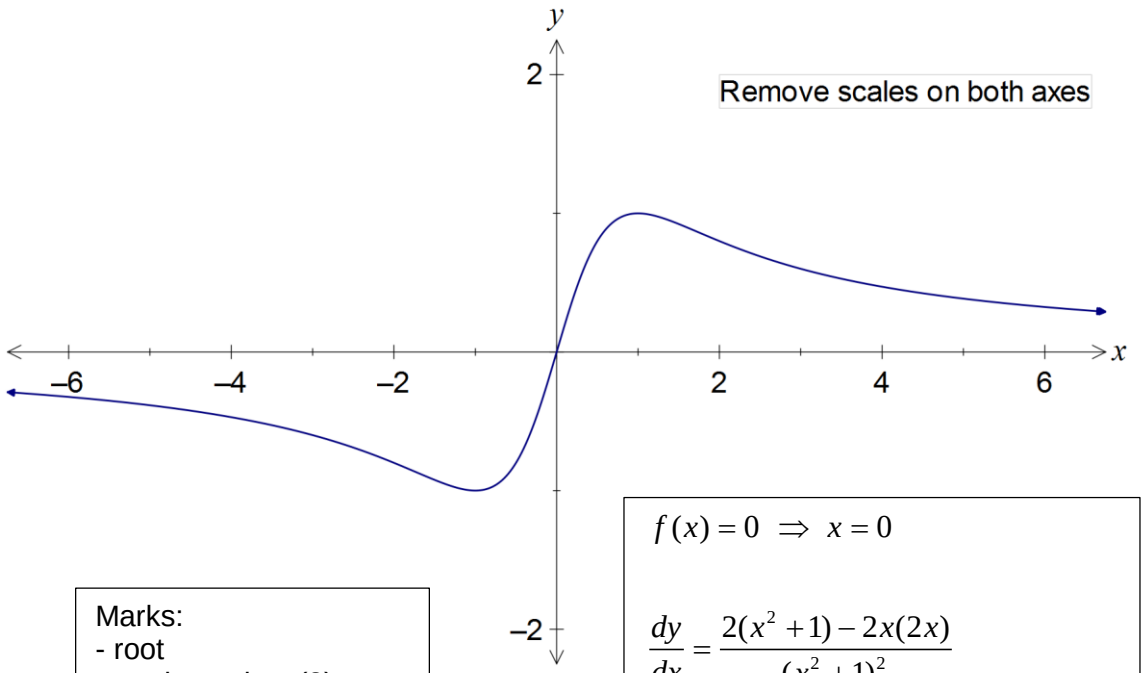
$$= \int 3\cos 2\theta\ d\theta$$

$$= \frac{3}{2}\sin 2\theta + c$$

Question 3 **(8 marks)**

The function f is defined by $f(x) = \dfrac{2x}{x^2+1}$.

(a) Sketch the graph of $y = f(x)$ on the axes below. (5 marks)



Remove scales on both axes

Marks:
- root
- turning points (2)
- x-axis as asymptote
- shape

$f(x) = 0 \Rightarrow x = 0$

$\dfrac{dy}{dx} = \dfrac{2(x^2+1) - 2x(2x)}{(x^2+1)^2}$

$\dfrac{dy}{dx} = 0 \Rightarrow 2 - 2x^2 = 0 \Rightarrow x = \pm 1$

$f(-1) = -1,\ f(1) = 1$

$x \to \pm\infty,\ y \to 0$

(b) State, with reasons, whether f is a one-to-one function. (1 mark)

No - f fails the horizontal line test (eg $y = 0.5$ is cut twice).

(c) If $g(x) = \dfrac{3}{x-a}$, determine the value(s) of the constant a so that the composite function $g \circ f(x)$ is defined for $x \in \mathbb{R}$. (2 marks)

$g \circ f(x)$ will be undefined if $f(x) - a = 0$, so require $a \neq f(x)$.

Range of $f(x)$ is $-1 \leq y \leq 1$, so $a > 1$ or $a < 1$.

See next page

Question 4 **(7 marks)**

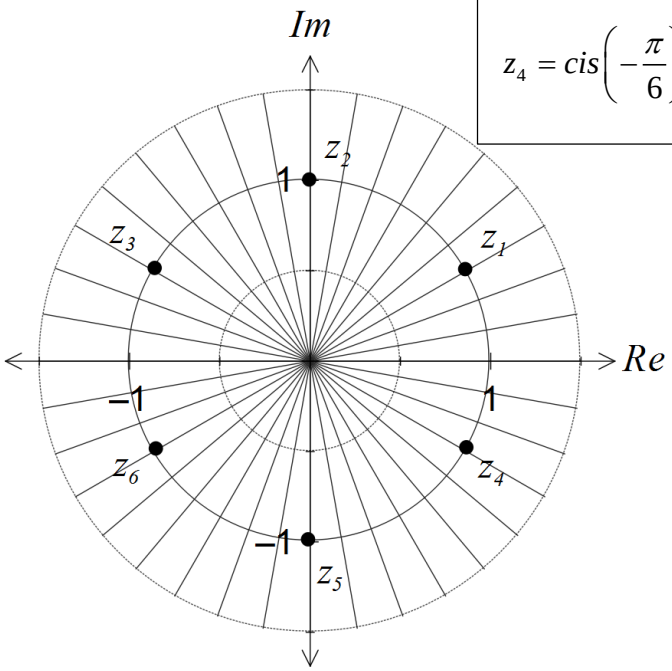
(a) Determine all solutions to the equation $z^6 = -1$, giving your answers in polar form and plotting all solutions on the Argand diagram below. (4 marks)

$z^6 = cis(\pi)$

$z = cis\left(\frac{\pi}{6} + \frac{2\pi n}{6}\right),\ n = ..., -1,\ 0,\ 1,\ 2,\ ...$

$z_1 = cis\left(\frac{\pi}{6}\right),\ z_2 = cis\left(\frac{\pi}{2}\right),\ z_3 = cis\left(\frac{5\pi}{6}\right)$

$z_4 = cis\left(-\frac{\pi}{6}\right),\ z_5 = cis\left(-\frac{\pi}{2}\right),\ z_6 = cis\left(-\frac{5\pi}{6}\right)$



(b) One of the solutions to $z^3 = c$ is $z = 3\sqrt{3} - 3i$. Determine all other solutions to the equation in Cartesian form. (3 marks)

$z_1 = 3\sqrt{3} - 3i = 6cis(-\frac{\pi}{6})$

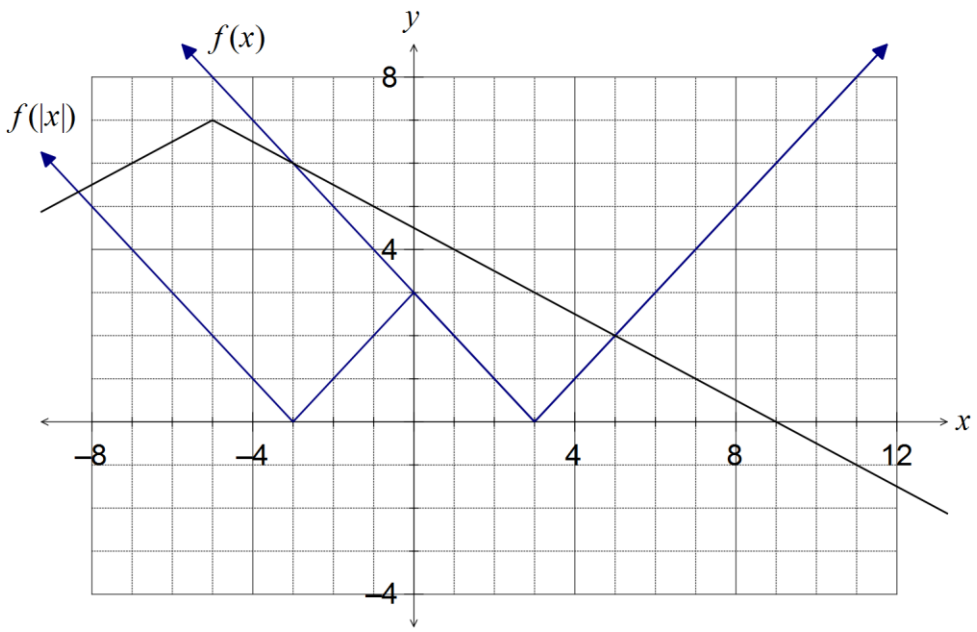
$z_2 = 6cis(-\frac{\pi}{6} + \frac{2\pi}{3}) = 6cis(\frac{\pi}{2})$
$= 6i$

$z_3 = 6cis(-\frac{\pi}{6} - \frac{2\pi}{3}) = 6cis(-\frac{5\pi}{6})$
$= -3\sqrt{3} - 3i$

See next page

Question 5 **(7 marks)**

Let $f(x) = |x-3|$ and $g(x) = 7 - \left| \frac{x}{2} + \frac{5}{2} \right|$. The graph of $y = g(x)$ is shown below.



(a) On the same axes, sketch and label the graphs of $y = f(x)$ and $y = f(|x|)$. (3 marks)

(b) Solve $|x-3| \geq 7 - \left| \frac{x}{2} + \frac{5}{2} \right|$. (1 mark)

$x \leq -3,\ x \geq 5$

(c) Given that the solution to $|ax+b| = 7 - \left| \frac{x}{2} + \frac{5}{2} \right|$ is $-5 \leq x \leq 9$, determine all possible values for the constants a and b. (3 marks)

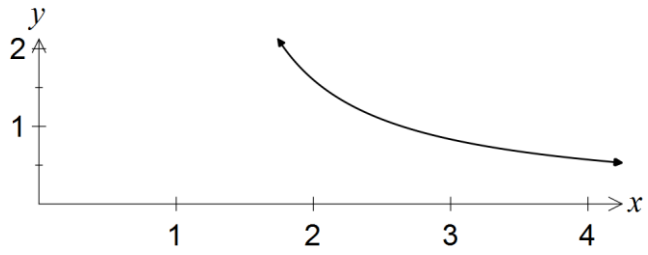
$ax + b = 7 - \frac{x}{2} - \frac{5}{2} \Rightarrow a = -0.5,\ b = 4.5$

$-ax - b = 7 - \frac{x}{2} - \frac{5}{2} \Rightarrow a = 0.5,\ b = -4.5$

See next page

Question 6 **(8 marks)**

Part of the graph of the function $f(x) = \dfrac{2x+4}{x^2+2x-3}$ is shown below.



(a) Use partial fractions to show that $f(x) = \dfrac{1}{2(x+3)} + \dfrac{3}{2(x-1)}$. (4 marks)

$$\frac{2x+4}{x^2+2x-3} = \frac{2x+4}{(x+3)(x-1)}$$

$$= \frac{A}{x+3} + \frac{B}{x-1}$$

$$Ax - A + Bx + 3B = 2x + 4$$

$$A + B = 2$$

$$-A + 3B = 4$$

$$B = \tfrac{3}{2},\ A = \tfrac{1}{2}$$

$$\frac{x+2}{x^2+2x-3} = \frac{1}{2(x+3)} + \frac{3}{2(x-1)}$$

(b) Show that the area under the graph of $y = f(x)$ between $x = 2$ and $x = 3$ is $\ln\left(\dfrac{4\sqrt{3}}{\sqrt{5}}\right)$. (4 marks)

$$\int_2^3 \frac{1}{2(x+3)} + \frac{3}{2(x-1)}\ dx = \left[\tfrac{1}{2}\ln|x+3| + \tfrac{3}{2}\ln|x-1|\right]_2^3$$

$$= \tfrac{1}{2}\ln 6 + \tfrac{3}{2}\ln 2 - \tfrac{1}{2}\ln 5 - \tfrac{3}{2}\ln 1$$

$$= \ln\frac{\sqrt{2}\sqrt{3}\times\sqrt{2}\sqrt{2}\sqrt{2}}{\sqrt{5}}$$

$$= \ln\frac{4\sqrt{3}}{\sqrt{5}}$$

See next page

Question 7 **(8 marks)**

Determine

(a) $\frac{d}{dx}\left(\tan^n(x)\right)$, where n is a positive integer. (2 marks)

$$\begin{aligned}\frac{d}{dx}\left(\tan^n(x)\right) &= \frac{d}{dx}\left(\left(\tan x\right)^n\right)\\ &= n\times\frac{d}{dx}\tan x\times\left(\tan x\right)^{n-1}\\ &= n\cdot\sec^2(x)\cdot\tan^{n-1}(x)\end{aligned}$$

(b) $\int \tan x\ dx$. (2 marks)

$$\int\frac{\sin x}{\cos x}dx = -\ln|\cos x|+c$$

$$NB:\ -\ln|\cos x| = \ln\left|\frac{1}{\cos x}\right| = \ln|\sec x|$$

(c) $\int \tan^5(x)\ dx$, using the identity $\tan^2\theta = \sec^2\theta - 1$ and your previous results. (4 marks)

$$\begin{aligned}\int\tan^5(x)\ dx &= \int\tan^3 x(\sec^2 x-1)\ dx\\ &= \int\tan^3 x\cdot\sec^2 x\ dx-\int\tan x(\sec^2 x-1)\ dx\\ &= \int\tan^3 x\cdot\sec^2 x\ dx-\int\tan x\cdot\sec^2 x\ dx+\int\tan x\ dx\\ &= \frac{1}{4}\tan^4 x-\frac{1}{2}\tan^2 x-\ln|\cos x|+c\end{aligned}$$

End of questions

Additional working space

Question number: _____

Additional working space

Question number: _____